SUBSTITUTION DIAGRAM OF ACOUSTIC CYLINDRICAL WAVEGUIDE INCLUDING VISCOSITY OF GAS INSIDE THE WAVEGUIDE

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Abstract: It is a common knowledge that the section of acoustic cylindrical waveguide can be described and replaced by the two-port network in the form of T-network or Π-network, on the basis of electroacoustic analogy with the homogeneous transmission lossless line. However, this replacement does not consider viscosity of gas inside the waveguide (losses inside the waveguide), which leads to incorrect determination of air column resonance frequencies in the cylindrical waveguide (in comparison with experimental measuring) as well as in the incorrect determination of the input impedance magnitude of the cylindrical waveguide. This contribution deals with the replacement of acoustic cylindrical waveguide, which includes the viscosity of gas inside the waveguide.

1. Introduction

Substitution diagrams are used with the convenience in the solution of the acoustic systems, because they convert the solution of these systems to the solution of the electric circuits thereby they often simplify significantly the solutions.

The comparisons of measured (measuring system BIAS [1]) and calculated resonance frequencies of oscillating air column inside cylindrical tubes with one open end and the second closed show, that the resonance frequencies calculated by the derived relation from the solution of the substitution diagram of the acoustic cylindrical waveguide (tube), do not correspond to the measured frequencies. This inaccuracy is mainly caused by the consideration, that inside the waveguide no losses occur, which therefore are not included in the substitution diagram. In fact, the viscothermal losses especially occur inside the waveguide, which are necessary to include to the substitution diagram of the cylindrical waveguide with respect to the improvement of the calculation resonance frequencies accuracy.

2. Lossless cylindrical waveguide

Acoustic cylindrical waveguide can be described by the substitution diagram in the form of the two-port network with the distributed elements, see **Fig. 1,** on the basis of the electroacoustic analogy with the homogenous transmission lossless line, on assumptions, that the losses inside the waveguide are not considered (the gas inside the waveguide is considered to be ideal gas) and a plane wave is propagating through the waveguide.

$$\hat{Z}_1 = \hat{Z}_2 = \frac{j\rho_0 c_0}{\pi R^2}\tan\left(\frac{kl}{2}\right) \tag{1}$$

$$\hat{Z}_3 = \frac{\rho_0 c_0}{\pi R^2}\frac{1}{j\sin(kl)} \tag{2}$$

$$\hat{Z}_1^* = \hat{Z}_2^* = \frac{\rho_0 c_0}{\pi R^2}\frac{\cot g\left(\frac{kl}{2}\right)}{j} \tag{3}$$

$$\hat{Z}_3^* = \frac{j\rho_0 c_0}{\pi R^2}\sin(kl) \tag{4}$$

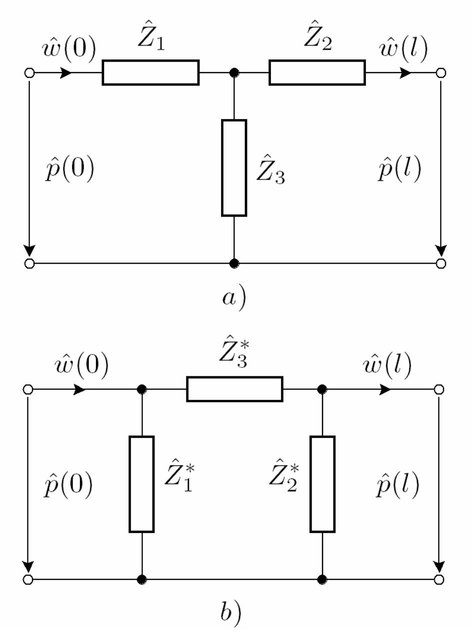


Figure 1. The substitution of the lossless cylindrical waveguide by the T-network (a), Π-network (b).

R is the radius of the tube, ρ_0 is the density of air and $k = \omega/c_0$ is the real wave number, where c_0 is the speed of the sound in free space (air).

3. Losses inside waveguide

Inside the waveguide the viscothermal losses occur by the effect of viscosity and thermal conductivity of the gas (air).
The section of the acoustic cylindrical waveguide can be described by means of substitution diagram with the distributed elements, see **Fig. 2**, also on the basis of electroacoustic analogy with the transmission line.

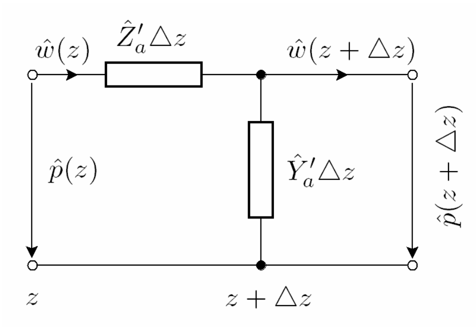


Figure 2. The substitution of the acoustic cylindrical waveguide section.

But $\hat{Z}_a'$ is the acoustic (series) impedance of the cylindrical waveguide per unit length of the waveguide, with the included viscosity losses and $\hat{Y}_a'$ is the acoustic (shunt) admittance of the cylindrical waveguide per unit length of the waveguide, with the included thermal losses. This substitution diagram (**Fig. 2**) is described by the equations (5) and (6).

$$\hat{p}(z) = \hat{w}(z)\hat{Z}_a'\Delta z + \hat{p}(z+\Delta z) \qquad (5)$$

$$\hat{w}(z) = \hat{p}(z+\Delta z)\hat{Y}_a'\Delta z + \hat{w}(z+\Delta z) \qquad (6)$$

After the modification of these equations and for $\Delta z \rightarrow 0$ we get the wave equation (7) with the complex wave number k^*, in which the viscothermal losses inside the waveguide are included, see relation (8).

$$\frac{d^2\hat{p}(z)}{dz^2} + k^{*2}\hat{p}(z) = 0 \qquad (7)$$

$$k^* = \sqrt{-\hat{Z}_a'\hat{Y}_a'} = \frac{\omega}{c} - j\alpha \qquad (8)$$

3.1. Effect of viscosity and thermal conduction

Acoustic particles of the gas (inside the waveguide) adjacent to the wall of the waveguide are inactive (at a motion of the gas) by the influence of an adhesion to the wall of the waveguide. In consequence of that, the acoustic particle velocity (of the gas) inside the waveguide changes with the distance from the wall of the waveguide, see **Fig. 3**. The friction among adjacent acoustic particles (in the radial direction) occurs with respect to this change of velocity. There is a viscous stress among adjacent acoustic particles (in the radial direction), which is proportional to the viscosity of the gas and to the change of acoustic particle velocity of the gas. The layer of the gas, in which the acoustic particle velocity change (of the gas) is high, is called the boundary layer, see **Fig.3**. Right in this layer the viscous stress asserts among adjacent acoustic particles of the gas (in the radial direction).

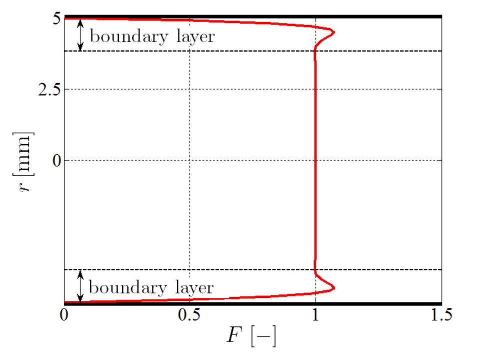


Figure 3. The course of the function F (velocity profile) indicating the distribution of the acoustic particle velocity of the air in the tube cross-section (in case that R = 0.005m and f = 100Hz).

The motion of incompressible gas with the viscosity is described by Navier-Stokes equation [2]. In cylindrical coordinates the Navier-Stokes equation has a form (on assumption of the laminar flow of the gas inside the tube, harmonic changes of the acoustic velocity and the pressure, considering acoustic particle velocity of the gas only in the direction of (axial) z axis of the (cylindrical) waveguide ($\vec{v} = \vec{e}_z v_z + \vec{e}_r 0 + \vec{e}_\varphi 0$) and considering, the component of the acoustic particle velocity v_z not depending on φ and z coordinates):

$$\mu\left(\frac{\partial^2\hat{v}_z}{\partial r^2} + \frac{1}{r}\frac{\partial\hat{v}_z}{\partial r}\right) = \frac{\partial\hat{p}_z}{\partial z} + \rho_0\frac{\partial\hat{v}_z}{\partial t}, \qquad (9)$$

where r is the distance from the z axis of the waveguide and μ is the viscosity of the gas inside the waveguide [3]. The solution of this equation is relation (10), which indicates the acoustic particle velocity of the gas depending on frequency and on the distance from the axis of the (cylindrical) waveguide [3].

$$\hat{v}_z(f,r) = \frac{-F(f,r)}{j\omega\rho_0}\frac{\partial\hat{p}_z}{\partial z} \qquad (10)$$

The function F indicates the distribution of the acoustic particle velocity of the gas in the cross section of the cylindrical waveguide, so it indicates so-called velocity profile, see **Fig. 3**. It is determined by the relation:

$$F = \frac{J_0(ar)}{J_0(aR)} - 1, \quad where \quad a = \sqrt{\frac{-j\omega\rho_0}{\mu}} \qquad (11)$$

Acoustic volume particle velocity is determined by the relation:

$$d\hat{w}_z = 2\pi r dr \hat{v}_z . \qquad (12)$$

By the integration of this relation we will get:

$$\hat{w}_z = \frac{-\partial\hat{p}_z}{\partial z}\frac{\pi R^2}{j\omega\rho_0}\frac{J_2(aR)}{J_0(aR)} . \qquad (13)$$

Then the acoustic impedance ($\hat{Z}'_a$) of the cylindrical waveguide per unit length of the waveguide is determined by the relation:

$$\hat{Z}'_a = \frac{-j\omega\rho_0}{\pi R^2}\frac{J_0(aR)}{J_2(aR)} . \qquad (14)$$

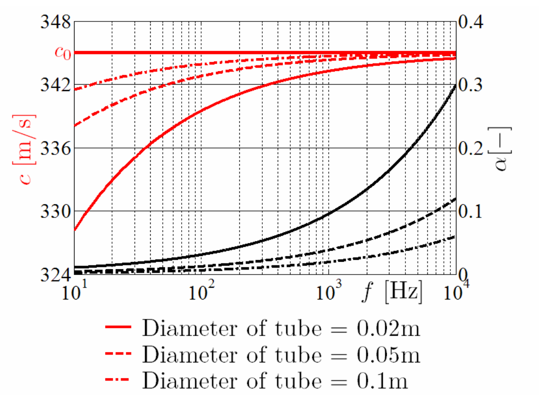


Figure 4. The courses of the attenuation coefficient (α) and the phase speed of the sound propagation (c) in the air inside the cylindrical waveguide (tube) in dependence on the frequency.

Due to very small acoustic particle velocity of the gas at the wall of the waveguide the process of the gas happening inside the tube can not be considered as purely adiabatic. There the thermal conduction among adjacent acoustic particles occurs.

Owing to the effect of the viscosity and the thermal conduction, the sound inside the tube propagates the phase speed c (not speed c_0), see **Fig. 4**, which depends not only on climatic conditions, but also on the frequency, radius of the tube, the viscosity and the coefficient of the thermal conduction, see relations (8), (14), (17).

The relation for the acoustic (shunt) admitance of the cylindrical waveguide (tube) we get from the solution of the ideal gas energy equation to describe by the gas temperature T (15) and from the differentiated approximated equation of ideal gas (16).

$$j\omega\rho_0 C_p\hat{T} = \kappa\Delta\hat{T} + j\omega\hat{p} \qquad (15)$$

$$V_T\hat{p} + p_0\hat{\Xi} = \rho_0 V_T(C_p - C_v)\overline{T} \qquad (16)$$

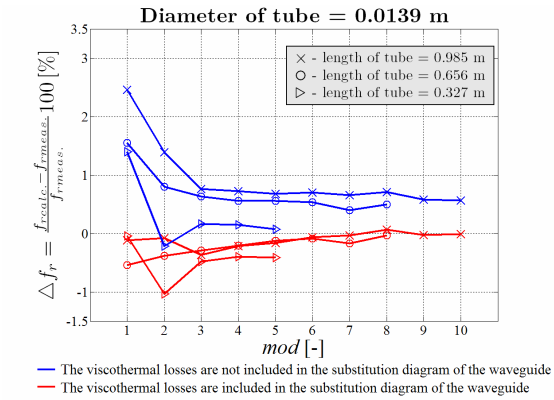
$\overline{T}$ is the average thermodynamic temperature, κ is the coefficient of the thermal conduction, C_p is the specific heat of air at constant pressure, C_v is the specific heat of air at constant volume, V_T is the static volume of the waveguide (tube), p_0 is the barometric pressure and Ξ is the volume translation. Then the acoustic admittance ($\hat{Y}'_a$) of the cylindrical waveguide per unit length of the waveguide is determined by the relation:

$$\hat{Y}'_a = \frac{j\omega\pi R^2}{\rho_0 c_0{}^2}\left(\gamma + (\gamma-1)\frac{J_2(bR)}{J_0(bR)}\right), \quad where \quad b = \sqrt{\frac{-j\omega\rho_0 P_r}{\mu}} = a\sqrt{P_r} . \qquad (17)$$

γ is the specific-heat ratio and $P_r = \sqrt{\mu C_p/\kappa}$ is the Prandtl number.

4. Conclusion

The effect of the viscothermal losses, which are included in the substitution diagram of the cylindrical waveguide, is shown in **Fig. 5**. In this figure there is made the comparison of the measured (measuring system BIAS) and calculated (the calculation was done by means of the substitution diagram of the acoustic cylindrical waveguide) resonance frequencies of oscillating air column inside the cylindrical tubes with one open end and the second closed. From the **Fig. 5**, we can see, that the smallest differences (calculated resonance frequencies from measured, measured only 10 modes) are obtained, in case, that the viscothermal losses are included to the substitution diagram of the acoustic cylindrical waveguide. Further, we can see as well, that viscothermal losses assert, above all, at the tubes with the small cross-section and on the low frequencies.



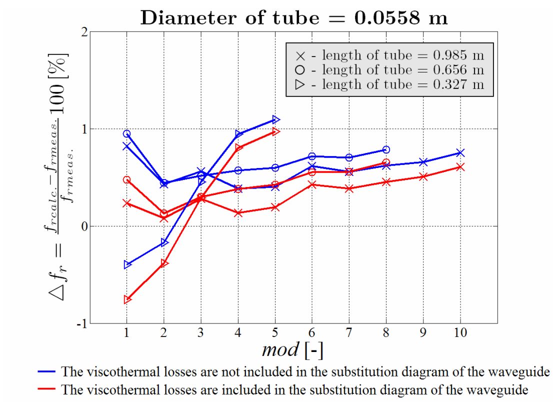


Figure 5. The comparison of the calculated and the measured resonance frequencies of the oscillating air column inside the cylindrical tubes (with one open end and the second closed) with the diameters 0.0139m, 0.0558m and the lengths 0.985m, 0.656m and 0.327m. The calculation was done by means of the substitution diagram of the acoustic cylindrical waveguide. In the calculation the effect of the surroundings outside the tube on the open end of the tube was considered as well. This effect was simulated by means of the radiation impedance of the equivalent pulsative sphere with the radius $R_{equ.}=R/\sqrt{2}$ [4].

We can conclude, that the viscothermal losses inside the waveguide can be described by the **complex** wave number, see relation (8), which replaces the **real** wave number in the wave equation for lossless cylindrical waveguide, see relation (7). The substitution diagram of the cylindrical waveguide, which includes the viscothermal losses inside the waveguide, is possible to be described again by the two-port network in the form of T-network or Π-network with the distributed elements, see **Fig. 1**. Above all in these elements the real wave number k is replaced by the complex wave number k^*, see relations (18) to (21).

$$\hat{Z}_1 = \hat{Z}_2 = \frac{j\rho_0 c_0}{\pi R^2}\tan\left(\frac{k^* l}{2}\right) \qquad (18)$$

$$\hat{Z}_3 = \frac{\rho_0 c_0}{\pi R^2}\frac{1}{j\sin(k^* l)} \qquad (19)$$

$$\hat{Z}_1^* = \hat{Z}_2^* = \frac{\rho_0 c_0}{\pi R^2}\frac{\mathrm{cot\,g}\left(\frac{k^* l}{2}\right)}{j} \qquad (20)$$

$$\hat{Z}_3^* = \frac{j\rho_0 c_0}{\pi R^2}\sin\left(k^* l\right) \qquad (21)$$

Other losses, of course, occur inside the waveguide, e.g. the losses due to the relaxation processes. These other losses are not considered.

Acknowledgements

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